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On Minimal Covolume Hyperbolic Lattices

Ruth Kellerhals

Department of Mathematics, University of Fribourg, CH-1700 Fribourg, Switzerland; ruth.kellerhals@unifr.ch

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Abstract: We study lattices with a non-compact fundamental domain of small volume in hyperbolic space $\mathbb{H}^n$. First, we identify the arithmetic lattices in $\mathrm{Isom}^+\,\mathbb{H}^n$ of minimal covolume for even n up to 18. Then, we discuss the related problem in higher odd dimensions and provide solutions for $n = 11$ and $n = 13$ in terms of the rotation subgroup of certain Coxeter pyramid groups found by Tumarkin. The results depend on the work of Belolipetsky and Emery, as well as on the Euler characteristic computation for hyperbolic Coxeter polyhedra with few facets by means of the program *CoxIter* developed by Guglielmetti. This work complements the survey about hyperbolic orbifolds of minimal volume.

Keywords: hyperbolic lattice; cusp; minimal volume; arithmetic group; Coxeter polyhedron

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1. Introduction

Let $n \geq 2$ and consider a hyperbolic lattice, that is, a discrete group $\Gamma \subset \mathrm{Isom}\,\mathbb{H}^n$ whose orbit space or *orbifold* $Q = \mathbb{H}^n/\Gamma$ is of finite volume. By a celebrated result of Kazhdan and Margulis, the set of all volumes $\mathrm{vol}_n(Q)$ has a positive minimal element μ_n. In the work [1], we provided a survey about the values μ_n and the volume minimising n-orbifolds for dimensions n satisfying $2 \leq n \leq 9$ by taking into account (non-)compactness, orientability, arithmeticity, and dimension parity.

In this work, we consider only the volumes of non-compact or *cusped* hyperbolic n-orbifolds and study the corresponding volume spectrum

$$\mathcal{V}_n := \{\,\mathrm{vol}_n(Q) \mid Q = \mathbb{H}^n/\Gamma \quad \text{non-compact}\}$$

with minimal element ν_n. The set $\mathcal{V}_n$ contains the proper subset $\mathcal{V}_n^a$ of volumes of orientable quotient spaces of $\mathbb{H}^n$ by arithmetic lattices in $\mathrm{Isom}^+\,\mathbb{H}^n$ with corresponding minimal element ν_n^a. By deep results of Belolipetsky and Emery (see [2–5]), the values ν_n^a are explicitly known for $n \geq 4$. Our aim is to describe the hyperbolic lattices whose covolumes equal ν_n^a for $n \geq 10$.

In this context, hyperbolic lattices generated by finitely many reflections in hyperplanes of $\mathbb{H}^n$, called *hyperbolic Coxeter groups*, are of particular interest (see Section 2.3). In fact, for $n \leq 9$, the smallest covolume hyperbolic Coxeter n-simplex groups (generated by $n+1$ reflections) are all arithmetic and yield the unique non-compact volume minimisers (up to a factor of two, in the exceptional case $n = 7$; for references, see [1], Section 3). In this way, the values ν_n and ν_n^a could be entirely specified. However, in $\mathrm{Isom}\,\mathbb{H}^n$, cofinite Coxeter simplex groups do not exist for $n \geq 10$ and, apart from Borcherds' example [6] for $n = 21$, nothing is known about the existence of cofinite hyperbolic Coxeter groups for $n \geq 20$.

In the sequel of our commensurability classification of Coxeter pyramid groups with $n+2$ generators existing up to dimension 17 (see [7]), Guglielmetti [8] developed the software program *CoxIter* testing various properties such as arithmeticity and providing invariants such as the Euler characteristic of a hyperbolic Coxeter group. In Section 2.6, we give several instructive examples. By a result of Emery [9], the covolume of the single Coxeter pyramid group $\Gamma_* \subset \mathrm{Isom}\,\mathbb{H}^{17}$ with Coxeter graph given by Figure 1 yields the minimal value among *all* ν_n^a for $n \geq 2$ (see also Section 3.1.2).

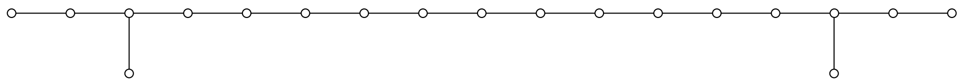


Figure 1. The Coxeter pyramid $P_* \subset \mathbb{H}^{17}$.

Based on these facts, we are able to identify the orientable cusped arithmetic hyperbolic n-orbifolds as orbit spaces by the action of certain hyperbolic Coxeter groups for the even dimensions n with $10 \leq n \leq 18$ and for the odd dimensions $n = 11$ and $n = 13$ (the proof for $n = 13$ is based on a combinatorial argument due to S. Tschantz [10]). The results are presented in Proposition 1, Proposition 2 and Proposition 3 of Section 3.1. The work ends with a couple of remarks about the mysterious case of dimension $n = 15$ and the non-arithmetic case.

2. Hyperbolic Lattices with Parabolic Elements

2.1. Hyperbolic n-Space

Denote by $\mathbb{X}^n$ one of the standard geometric spaces given by either the Euclidean space $\mathbb{E}^n$, the standard sphere $\mathbb{S}^n$, or hyperbolic space $\mathbb{H}^n$. View each space $\mathbb{X}^n$ in the context of a linear space equipped with a suitable bilinear form $\langle \cdot, \cdot \rangle$. In particular, $\mathbb{H}^n$ is interpreted as a connected component $\mathcal{H}^n$ of the two-sheeted hyperboloid in $\mathbb{R}^{n+1}$ according to:

$$\mathcal{H}^n = \{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \langle x, x \rangle_{n,1} = x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = -1 \, , \, x_{n+1} > 0 \} \, .$$

In this picture, the isometry group $\operatorname{Isom} \mathbb{H}^n$ is isomorphic to the group $\mathrm{PO}(n,1)$ of positive Lorentzian matrices leaving invariant the product $\langle \cdot, \cdot \rangle_{n,1}$ and $\mathcal{H}^n$ (cf. [11] Chapter 3). By passing to the upper half space model $\mathcal{U}^n$ of $\mathbb{H}^n$ in $\mathbb{E}^n_+$, the line element and the volume element are given by:

$$ds^2 = \frac{dx_1^2 + \ldots + dx_n^2}{x_n^2} \quad , \quad \mathrm{dvol}_n = \frac{dx_1 \cdot \ldots \cdot dx_n}{x_n^n} \, .$$

Furthermore, a horizontal hyperplane $S_\infty(a) = \{ x \in \mathcal{U}^n \mid x_n = a \}$, where $a > 0$, carries a Euclidean metric up to a distortion factor $1/a^2$. Such a subset is called a *horosphere* based at ∞ and bounds the *horoball* $B_\infty(a) = \{ x \in \mathcal{U}^n \mid x_n > a \}$ above it.

2.2. Cusped Hyperbolic Orbifolds of Finite Volume

Consider a discrete subgroup $\Gamma \subset \operatorname{Isom} \mathbb{H}^n$ with fundamental domain $D \subset \mathbb{H}^n$, and suppose that Γ is *of finite covolume* or *cofinite* for short. This means that the volume of the orbifold $Q = \mathbb{H}^n/\Gamma$, as given by the volume of D, is finite, and we call Γ a *hyperbolic lattice*, for short. In the sequel we are particularly interested in lattices Γ giving rise to non-compact or *cusped* orbifolds $\mathbb{H}^n/\Gamma$. Such a group Γ contains at least one non-trivial subgroup Γ_q of parabolic type whose elements stabilise a point $q \in \partial\mathbb{H}^n$, say (see [12], Section 3.1). Associated to the fixed point q at infinity is a *cusp neighborhood* $C_q \subset V$ which is an embedded subset given by the quotient B_q/Γ_q of a Γ_q-precisely invariant horoball B_q in $\mathbb{H}^n$. The action of Γ_q on the boundary horosphere ∂B_q is by Euclidean isometries and with a compact fundamental domain so that Γ_q can be interpreted as a *crystallographic* subgroup of $\operatorname{Isom} \mathbb{E}^{n-1}$. By Bieberbach's theory, any crystallographic group in $\operatorname{Isom} \mathbb{E}^{n-1}$ contains a finite index translational lattice of rank $n-1$. For $n \leq 9$, it is known (see [13,14] and also Section 3) that small volume cusped hyperbolic n-orbifolds are intimately related to dense lattice packings in $\mathbb{E}^{n-1}$.

2.3. Hyperbolic Coxeter Polyhedra and Discrete Reflection Groups with Few Generators

Consider a geometric polyhedron $P \subset \mathbb{X}^n$, that is, P is an n-dimensional convex polyhedron of finite volume in $\mathbb{X}^n$, bounded by $N \geq n+1$ hyperplanes $H_i \subset \mathbb{X}^n$ with unit normal vector e_i directed away from, say, P. Denote by $G(P) = (\langle e_i, e_j \rangle)_{1 \leq i,j \leq N}$, the Gram matrix of P. In [15,16], Vinberg developed a very satisfactory theory to conclude the existence and describe arithmetic, combinatorial, and metrical properties of P in terms of $G(P)$. There are explicit criteria by means of certain submatrices of $G(P)$ allowing one to decide whether P is compact or of finite volume. In this work, we are dealing mainly with non-compact hyperbolic polyhedra of finite volume, being the convex hull of finitely many ordinary points in $\mathbb{H}^n$ and at least one point in the boundary $\partial\mathbb{H}^n$ at infinity of $\mathbb{H}^n$. In fact, the fundamental group of a finite volume cusped hyperbolic n-orbifold admits a fundamental domain whose closure is a hyperbolic polyhedron and any of its parabolic subgroups stabilises a vertex at infinity of the fundamental polyhedron.

A geometric polyhedron $P \subset \mathbb{X}^n$ is a *Coxeter polyhedron* if all of its dihedral angles $\alpha = \alpha_F$ formed by pairs H, H' of intersecting hyperplanes in the boundary of P, and with associated ridge $F = H \cap H' \cap P$, are of the form π/m for an integer $m \geq 2$. Coxeter polyhedra arise in a natural way as building blocks in the context of regular polyhedra and as closures of fundamental regions of discrete reflection groups. Denote by $\Gamma = \Gamma(P)$ the group generated by the N reflections $s = s_H$ with respect to the hyperplanes H bounding the Coxeter polyhedron P; the group Γ is called the *geometric Coxeter group* associated to P. The group Γ is a discrete subgroup in $\operatorname{Isom}\mathbb{X}^n$ which admits a particularly simple presentation with relations satisfying $s^2 = 1$ and

$$(ss')^m = 1 \quad \text{for an integer } m = m(s,s') \geq 2\,, \tag{1}$$

for distinct generators $s = s_H$ and $s' = s_{H'}$, with hyperplanes H and H' intersecting in $\mathbb{X}^n$ along a ridge $H \cap H' \cap P$ where P has dihedral angle π/m.

Most conveniently, geometric Coxeter polyhedra of simple combinatorics (and Coxeter groups with few generators) are described by their *Coxeter graph* Σ. Each node ν in Σ corresponds to a hyperplane H (and the reflection $s = s_H$) and is joined to another node ν' by an edge with weight m if the corresponding dihedral angle, formed by their hyperplanes H, H' at the ridge F in P, equals $\alpha_F = \pi/m$. Usually, edges with a weight 2 are omitted and edges with weight 3 (resp. 4) are drawn as simple (resp. double edges). In the hyperbolic case, and for parallel hyperplanes H, H' intersecting in $\partial\mathbb{H}^n$, the nodes ν, ν' are connected by a bold edge; for hyperplanes H, H' disjoint in $\mathbb{H}^n$ and of hyperbolic distance l, the nodes ν, ν' are connected by a dotted edge (and sometimes marked with the weight l).

In contrast to the spherical and Euclidean cases, Coxeter polyhedra in $\mathbb{H}^n$ are classified only very partially. There is a complete list for hyperbolic Coxeter simplices, characterised by $N = n+1$, and they exist for $n \leq 9$, only. Hyperbolic Coxeter polyhedra with $N = n+2$ are classified and they exist for $n \leq 17$. Notice that examples of compact Coxeter polyhedra in $\mathbb{H}^n$ are known just for $n \leq 8$. In higher dimensions, there are only single examples in $\mathbb{H}^n$ for $n = 18, 19$, and 21 whose discovery is due to Kaplinskaja, Vinberg, and Borcherds, respectively. Notice that Coxeter polyhedra in $\mathbb{H}^n$ do not exist for $n > 995$. For a survey, we refer to information and references collected on the webpage of Felikson and Tumarkin [17].

As for even dimensions above 17, there are only the two Coxeter polyhedra P_{18} and P_{18}^L explicitly known (for more details, see Example 7). The polyhedron P_{18} exists in $\mathbb{H}^{18}$ and is non-compact and bounded by 37 hyperplanes only forming dihedral angles of $\pi/2$ and $\pi/3$. The polyhedron was discovered by Kaplinskaja and Vinberg [18] and is associated with the maximal reflection group Γ_{18} in the group $\mathrm{PO}(18,1;\mathbb{Z})$ of integral positive linear transformations leaving invariant the unimodular quadratic form $q_n = \langle x, x\rangle_{n,1}$ for $n = 18$. Observe that the quadratic forms q_n are *reflective* in the above sense, providing finite volume non-compact hyperbolic Coxeter polyhedra P_n, for $n \leq 19$. More precisely, the group $\mathrm{PO}(n,1;\mathbb{Z})$ is the automorphism group $\mathrm{PO}(\mathrm{I}_{n,1})$ of the odd unimodular Lorentzian lattice $\mathrm{I}_{n,1}$ with quadratic form q_n whose maximal reflection subgroup is of finite index equal to the order of the symmetry group $\mathrm{Sym}(P_n)$ of its (fundamental) Coxeter polyhedron P_n. The order of $\mathrm{Sym}(P_n)$ is different from 1 for $14 \leq n \leq 19$ and equal to 2 (resp. 4) for $n = 14, 15$ (resp. $n = 16, 17$), whereas the symmetric group S_n appears according $\mathrm{Sym}(P_{18}) \cong S_4$ and $\mathrm{Sym}(P_{19}) \cong S_5$. The results can be found in [18,19] and ([16] part II, Chapter 6, §2).

2.4. Arithmetic Hyperbolic Coxeter Groups

The group $\mathrm{PO}(18,1;\mathbb{Z})$ is a model of an *arithmetic* group, a notion which will not be explained in detail here (see [4,5], for example). One characterisation is —by using a result of Margulis (see [20], Theorem 10.3.5, for example) —that a lattice $G \subset \mathrm{Isom}\,\mathbb{H}^n$, $n \geq 3$, is non-arithmetic if and only if its *commensurator group* $\mathrm{Comm}(G)$ is discrete in $\mathrm{Isom}\,\mathbb{H}^n$ (and containing G with finite index). Here, the group $\mathrm{Comm}(G)$ is defined by:

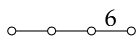
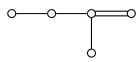
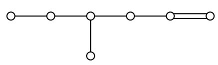
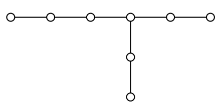
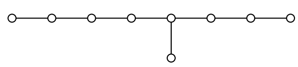
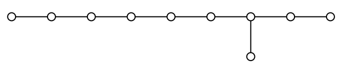
$$\mathrm{Comm}(G) = \{\, \gamma \in \mathrm{Isom}\,\mathbb{H}^n \mid G \cap \gamma G \gamma^{-1} \text{ has finite index in } G \text{ and } \gamma G \gamma^{-1} \,\}\,.$$

However, for hyperbolic Coxeter groups $\Gamma \subset \mathrm{PO}(n,1)$ such as Γ_{18}, Vinberg developed a very useful criterion for arithmeticity. This criterion simplifies drastically when the group Γ has a *non-compact* (fundamental) Coxeter polyhedron $P \subset \mathbb{H}^n$. Consider the Gram matrix $G = G(P)$ of P and form the matrix $H := 2 \cdot G$ with coefficients h_{ij} for $1 \leq i,j \leq N$. A *cycle* in H is a product of the type $h_{i_1 i_2} \cdot h_{i_2 i_3} \cdot \ldots \cdot h_{i_{k-1} i_k} \cdot h_{i_k i_1}$.

Theorem (Vinberg's Criterion)**.** *Let $P \subset \mathbb{H}^n$ be a non-compact hyperbolic Coxeter polyhedron with Coxeter group Γ and Gram matrix G. Then, Γ is arithmetic if and only if all the cycles of the matrix $2 \cdot G$ are rational integers.*

Example 1. *The non-compact hyperbolic Coxeter simplices with graphs Ξ_n , $2 \leq n \leq 9$, given in Table 1 are all arithmetic.*

Table 1. Some non-compact hyperbolic Coxeter simplices.

dim n	Ξ_n
2	
3	
4	
5	
6	
7	
8	
9	

Example 2. *The Coxeter polyhedron $P_* \subset \mathbb{H}^{17}$ given by the graph in Figure 1 is bounded by 19 hyperplanes and has precisely two vertices at infinity. It has the combinatorial type of a pyramid over a product of two (eight-dimensional) simplices. Coxeter polyhedra of this type have been classified by Tumarkin [21]. The polyhedron P_* yields an arithmetic reflection group Γ_*, as is easily checked by means of Vinberg's criterion above. The group Γ_* is the maximal reflection in the automorphism group $PO(II_{17,1})$ of the even unimodular Lorentzian lattice $II_{17,1}$. Due to the obvious two-fold symmetry of the graph, one can pass to the $\mathbb{Z}_2$-extension of the group Γ_*, which is arithmetic of half the covolume of Γ_*.*

2.5. The Euler Characteristic and the Covolume of a Hyperbolic Coxeter Group

Let $\Gamma \subset \mathrm{Isom}\,\mathbb{H}^n$ be a Coxeter group with presentation $\langle S \mid R \rangle$ according to (1) and fundamental polyhedron $P \subset \mathbb{H}^n$ of finite volume. Consider the *growth series*

$$f_S(x) = \sum_{\gamma \in \Gamma} x^{l_S(\gamma)} \tag{2}$$

where $l_S(\gamma)$ is the word length of $\gamma \in \Gamma$ with respect to the generating set S of Γ. Denote $\mathcal{F} = \{\, T \subset S \mid \Gamma_T < \Gamma \text{ finite} \,\}$ the set of all subsets T of S such that the group Γ_T generated by the elements in T is finite. Notice that the groups of type Γ_T are spherical Coxeter groups with finite growth series. In order to represent their growth polynomials, we use the standard notations $[k] := 1 + x + \cdots + x^{k-1}$, $[k,l] = [k] \cdot [l]$ and so on, and denote by $m_1 = 1, m_2, \ldots, m_t$ the exponents of the Coxeter group Γ_T (see [22], Section 9.7). For the list of irreducible finite Coxeter groups, see Table 2.

Table 2. Exponents and growth polynomials of irreducible finite Coxeter groups Γ_T.

Graph	Exponents	Growth polynomial $f_T(x)$
$G_2^{(m)}$	$1, m-1$	$[2, m]$
A_n	$1, 2, \ldots, n-1, n$	$[2, 3, \ldots, n, n+1]$
B_n	$1, 3, \ldots, 2n-3, 2n-1$	$[2, 4, \ldots, 2n-2, 2n]$
D_n	$1, 3, \ldots, 2n-5, 2n-3, n-1$	$[2, 4, \ldots, 2n-2] \cdot [n]$
F_4	$1, 5, 7, 11$	$[2, 6, 8, 12]$
E_6	$1, 4, 5, 7, 8, 11$	$[2, 5, 6, 8, 9, 12]$
E_7	$1, 5, 7, 9, 11, 13, 17$	$[2, 6, 8, 10, 12, 14, 18]$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$	$[2, 8, 12, 14, 18, 20, 24, 30]$
H_3	$1, 5, 9$	$[2, 6, 10]$
H_4	$1, 11, 19, 29$	$[2, 12, 20, 30]$

By a result of Steinberg [23], $f_S(x)$ is the power series of a rational function and satisfies the following important formula.

$$\frac{1}{f_S(x^{-1})} = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(x)}\,, \tag{3}$$

where $\Gamma_{\varnothing} = \{1\}$. For the Euler characteristic $\chi(\Gamma)$, one obtains, for any abstract Coxeter group Γ,

$$\chi(\Gamma) = \sum_{T \in \mathcal{F}} \frac{(-1)^{|T|}}{f_T(1)}\,. \tag{4}$$

In terms of the volume of P and therefore the quotient space $\mathbb{H}^n/\Gamma$, one deduces the following identity (see [24]).

$$\chi(\Gamma) = \begin{cases} \dfrac{(-1)^{\frac{n}{2}}\, 2\,\mathrm{vol}_n(P)}{\mathrm{vol}_n(\mathbb{S}^n)}\,, & \text{if } n \text{ is even}\,, \\ 0\,, & \text{if } n \text{ is odd}\,. \end{cases} \tag{5}$$

The formulas (3) and (5) are very useful when computing the volume of an *even-dimensional* hyperbolic Coxeter polyhedron. Since the list of irreducible finite Coxeter groups is known and comparatively short (see Table 2), this volume computation can be realised by a computer program.

2.6. The Computer Program CoxIter

By means of the computer program *CoxIter* designed by Guglielmetti [8] in 2015 (freely accessible online https://coxiter.rgug.ch/, https://coxiterweb.rafaelguglielmetti.ch/), different invariants of a Coxeter group Γ acting by reflections on $\mathbb{H}^n$ can be computed. The input are the dimension n and the Coxeter graph Σ with the number of its nodes and with the edge weights $m > 2$ (either 0 or -1 for parallel or disjoint hyperplanes, respectively). Then, the program *CoxIter* answers questions such as cocompactness, cofiniteness, arithmeticity, Euler characteristic and covolume of Γ, number of vertices at infinity, and the f-vector (with components f_i equal to the number of i-dimensional faces) of its Coxeter polyhedron P.

Example 3. *Consider the two Coxeter pyramids $P_{10} \subset \mathbb{H}^{10}$ and $P_{12} \subset \mathbb{H}^{12}$ with Coxeter graphs given by Figures 2 and 3. By the tools mentioned in Sections 2.4 and 2.5, one can check easily that the associated Coxeter groups Γ_{10} and Γ_{12} are arithmetic. By means of the webversion of CoxIter, one computes their Euler characteristic as being equal to $\chi(\Gamma_{10}) = -1/183936614400$ (see also [14], Appendix A3) and $\chi(\Gamma_{12}) = 691/382588157952000$ (see the output given in Figure 4).*

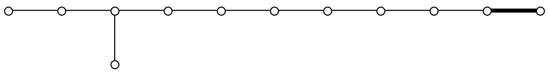


Figure 2. The Coxeter pyramid group $\Gamma_{10} \subset \operatorname{Isom} \mathbb{H}^{10}$ of covolume $\frac{\pi^5}{5431878144000}$.

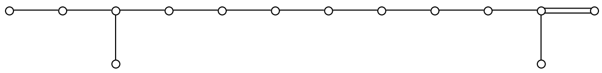


Figure 3. The Coxeter pyramid group $\Gamma_{12} \subset \operatorname{Isom} \mathbb{H}^{12}$ of covolume $\frac{691\,\pi^6}{62140685967360000}$.

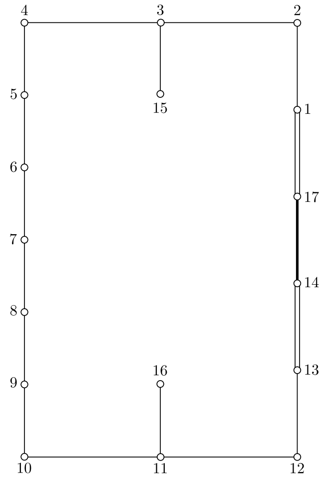
```
Input
    14 12
    vertices labels: 1 2 3 4 5 6 7 8 9 10 11 12 13 14
    1 2 3
    2 3 3
    3 4 3
    3 13 3
    4 5 3
    5 6 3
    6 7 3
    7 8 3
    8 9 3
    9 10 3
    10 11 3
    11 12 4
    11 14 3

Invariants
    Cocompact: no
    Cofinite: yes
    f-vector: (37, 234, 786, 1749, 2793, 3312, 2958, 1992, 1000, 364, 91, 14, 1)
    Number of vertices at infinity: 2
    Euler characteristic: 691/382588157952000
    Covolume: pi^6 * 691/62140685967360000
```

Figure 4. Output of the webversion of *CoxIter* for the group Γ_{12}.

Example 4. *Consider the Coxeter group Γ_{14} generated by 17 reflections in $\operatorname{Isom} \mathbb{H}^{14}$ with graph given in Figure 5. It was discovered by Vinberg as being the maximal reflection subgroup of the group of units of the unimodular quadratic form q_{14} of signature $(14, 1)$. The program CoxIter yields the following information.*

```
17 14
1 2 3
2 3 3
3 4 3
4 5 3
5 6 3
6 7 3
7 8 3
8 9 3
9 10 3
10 11 3
11 12 3
12 13 3
13 14 4
14 17 0
17 1 4
11 16 3
3 15 3
```

Content of the text file



Corresponding Coxeter graph

Figure 5. Vinberg's hyperbolic lattice $\Gamma_{14} \subset \mathrm{Isom}\,\mathbb{H}^{14}$.

```
Information
    Cocompact: no
    Finite covolume: yes
    Arithmetic: yes
    f-vector: (94, 704, 2695, 6825, 12579, 17633, 19215, 16425, 11009,
    5733, 2275, 665, 135, 17, 1)
    Number of vertices at infinity: 5
    Alternating sum of the components of the f-vector: 0
    Euler characteristic: -87757/289236647411712000
    Covolume: pi^7 * 87757/3053593308436070400000
```

Example 5. *The Coxeter group $\Gamma'_{16} \subset \mathrm{Isom}\,\mathbb{H}^{16}$ with Coxeter graph given by Figure 6, also discovered by Vinberg, is the maximal reflection subgroup of the group of units of the unimodular quadratic form q_{16}. Here, CoxIter provides the following data (see also [8], Table 2).*

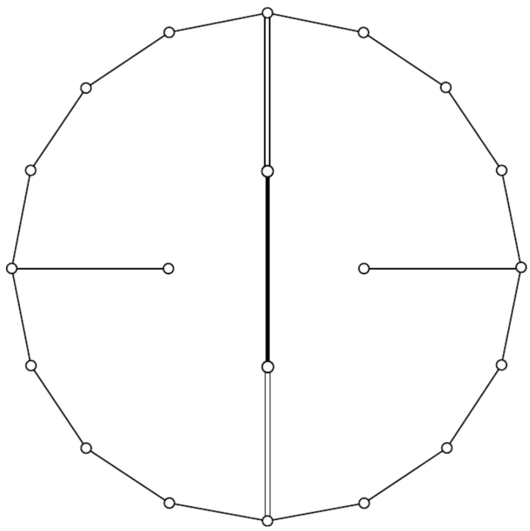


Figure 6. Vinberg's hyperbolic lattice $\Gamma'_{16} \subset \mathrm{Isom}\,\mathbb{H}^{16}$.

```
Information
    Cocompact: no
    Finite covolume: yes
    Arithmetic: yes
    f-vector: (325, 2804, 11914, 33164, 67410, 105462, 130646, 130062, 104670,
    68042, 35490, 14658, 4690, 1122, 189, 20, 1)
    Number of vertices at infinity: 12
    Alternating sum of the components of the f-vector: 0
    Euler characteristic: 642332179/2360171042879569920000
    Covolume: pi^8 * 642332179/1868799104762875084800000
```

Example 6. *The Coxeter group $\Gamma_{16} \subset \operatorname{Isom} \mathbb{H}^{16}$ with 19 generators and with Coxeter graph given by Figure 7 was discovered by Tumarkin [25]. It is distinguished by the fact that it represents the single top-dimensional cofinite hyperbolic Coxeter group in $\operatorname{Isom} \mathbb{H}^n$ with $n+3$ generators. Furthermore, Γ_{16} is arithmetic and CoxIter provides the following further details (see also [8], Table 2).*

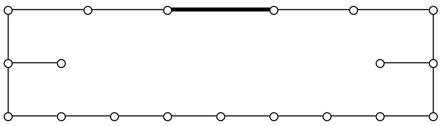


Figure 7. Tumarkin's hyperbolic lattice $\Gamma_{16} \subset \operatorname{Isom} \mathbb{H}^{16}$

```
Information
    Cocompact: no
    Cofinite: yes
    f-vector: (128, 1087, 4768, 14000, 30352, 50960, 67960, 72908, 63204,
    44200, 24752, 10948, 3740, 952, 170, 19, 1)
    Number of vertices at infinity: 3
    Euler characteristic: 2499347/2360171042879569920000
    Covolume: pi^8 * 2499347/1868799104762875084800000
```

Example 7. *In [26] Section 7, Vinberg constructed a particular quadratic form by considering the lattice $L = L_0 \oplus \mathbb{Z}e \subset (\mathbb{R}^{19}, \langle \cdot, \cdot \rangle_{18,1})$, where $L_0 := II_{17,1}$ is the even unimodular quadratic lattice of signature $(17,1)$, and e is a long root of norm two. Recall that the automorphism group of the lattice L_0 yields the standard form q_{17}, which is reflective with maximal reflection subgroup Γ_{17} of index 2 (see Section 2.3 and Figure 1). By means of an algorithm developed earlier by Vinberg, he proved that the lattice L yields a reflective quadratic form as well, and this by construction of a finite volume Coxeter polyhedron $P_{18}^L \subset \operatorname{Isom} \mathbb{H}^{18}$ with explicit description of the Coxeter graph. The associated arithmetic Coxeter group Γ_{18}^L is given by Figure 8. In particular, by applying Guglielmetti's program CoxIter to the Coxeter group $\Gamma_{18}^L \subset \operatorname{Isom} \mathbb{H}^{18}$, which is generated by 24 reflections, the Euler characteristic equals $-\frac{109638854849}{2260099790661476155392000}$. Hence, the covolume can be computed as follows (see [8] Table 5).*

$$\operatorname{covol}_{18}(\Gamma_{18}^L) = \frac{691 \times 3617 \times 43867}{2^{23} \times 3^{16} \times 5^6 \times 7^4 \times 11^2 \times 13^2 \times 17^2 \times 19} \pi^9 \approx 2.148561 \times 10^{-15}. \tag{6}$$

In an earlier work, together with Kaplinskaja, Vinberg [18] used the algorithm mentioned above to prove that the unimodular quadratic forms q_n are also reflective for $n = 18$ and $n = 19$ (while this is not the case for $n \geq 20$). Furthermore, they provided the corresponding Coxeter graphs. By CoxIter, Guglielmetti computed the covolume of the Coxeter group $\Gamma_{18} \subset \operatorname{Isom} \mathbb{H}^{18}$, related to q_{18}, which is generated by 37 reflections, and found that $\chi(\Gamma_{18}) = -\frac{109638854849}{1482580623111880900608000000}$ so that the covolume is $\approx 2.204424 \times 10^{-12}$ (see [8] Table 4).

Observe that the numerator $109638854849 = 691 \times 3617 \times 43867$ *of* $\chi(\Gamma_{18})$ *is identical to the one of* $\chi(\Gamma_{18}^{L})$*, a direct consequence of the formula (4) and the fact that both Coxeter graphs have integer weights 2 and 3, only.*

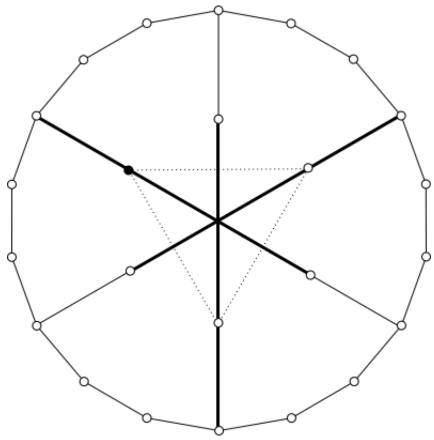


Figure 8. Vinberg's hyperbolic lattice $\Gamma_{18}^{L} \subset \mathrm{Isom}\,\mathbb{H}^{18}$.

Remark 1. *In the sequel of the two hyperbolic Coxeter groups* Γ_{18} *and* Γ_{18}^{L}*, there are, up until now, no such reflection groups in* $\mathrm{Isom}\,\mathbb{H}^{2m}$ *known with explicit presentation and covolume for* $m > 9$*. In [27], Ratcliffe and Tschantz considered arithmetic n-space forms* M_k^n *given as quotients of* $\mathbb{H}^n$ *by the principal congruence subgroups* $\Delta_k \subset PO(n,1;\mathbb{Z})$ *of level* $k \geq 1$*. The spaces* M_k^n *are smooth manifolds for* $k \geq 3$*. The orbifolds* M_1^n*, which are closely related to the quadratic form* q_n*, are well understood for* $n \leq 19$ *(see examples above). By means of Theorems 6, 8 and 25 in [27],Ratcliffe and Tschantz provide an explicit volume formula for* M_p^n *for all* $n \geq 2$ *and odd prime numbers* p*, by exploiting a result of Siegel related to the case* M_1^n*.*

3. Minimal Volume Cusped Hyperbolic Orbifolds

Let $\Gamma \subset \mathrm{Isom}\,\mathbb{H}^n$ be a lattice with fundamental polyhedron $P \subset \mathbb{H}^n$ such that the n-orbifold $\mathbb{H}^n/\Gamma$ is non-compact. This implies that P is the convex hull of finitely many vertices, with at least one vertex q belonging to $\partial\mathbb{H}^n$, whose stabiliser $\Gamma_q \subset \Gamma$ is a crystallographic group (see Section 2.2). Consider the volume spectrum:

$$\mathcal{V}_n = \{\, \mathrm{vol}_n(Q) \mid Q = \mathbb{H}^n/\Gamma \quad \text{non-compact}\}$$

of all cusped hyperbolic n-orbifolds together with its minimal element $\nu_n \in \mathcal{V}_n$ for each $n \geq 2$.

In [12], a universal lower volume bound for cusped hyperbolic n-manifolds has been established that also holds in the singular case (see [28] Section 2, [13] Section 2 and [14] Chapter 5). It provides a lower bound for ν_n in terms of (lattice) packing densities and orders of maximal point groups.

More precisely, denote by vol_k° the Euclidean k-volume functional and by $B(r)$ a Euclidean r-ball. Let φ_k be the maximal point group order of elements in a fixed $\mathbb{Q}$-class of maximal, finite, absolutely irreducible subgroups of $\mathrm{GL}(k,\mathbb{Z})$, and let δ_k be the maximal lattice packing density in Euclidean k-space. In particular, one has that $\varphi_{24} = 2^{22} \times 3^9 \times 5^4 \times 7^2 \times 11 \times 13 \times 23 = 8315553613086720000$, which is equal to twice the order of the Conway group Co_1, and that $\delta_8 = \pi^4/384 \approx 0.25367$ and $\delta_{24} = \pi^{12}/12! \approx 0.00193$ (see [29]). Notice that for $n \leq 8$, the densest lattice packings are known and intimately related to root lattices. Moreover, by a recent fundamental result of Viazovska [30], the E_8 root lattice yields the densest sphere packing in $\mathbb{E}^8$, leading to a proof of similar flavour, by Cohn, Kumar, Miller, Radchenko and Viazovska [31], showing that the Leech lattice is an optimal sphere packing in $\mathbb{E}^{24}$.

The value $d_n(\infty)$ denotes the *simplicial horoball density*, that is, the density of $n+1$ horoballs based at the vertices of an ideal regular simplex in $\mathbb{H}^n$. By means of the volume ω_n of an ideal regular

n-simplex with its representation as an infinite series given by Milnor, $d_n(\infty)$ can be expressed as follows (see [12], Theorem 2.1).

$$d_n(\infty) = \frac{n+1}{n-1} \cdot \frac{n}{2^{n-1}} \cdot \prod_{k=2}^{n-1} \left(\frac{k-1}{k+1} \right)^{\frac{n-k}{2}} \cdot \frac{1}{\omega_n} = \frac{n+1}{n-1} \cdot \frac{\sqrt{n}}{2^{n-1}} \cdot \frac{\prod_{k=2}^{n-1} \left(\frac{k-1}{k+1}\right)^{\frac{n-k}{2}}}{\sum_{k=0}^{\infty} \frac{\beta(\beta+1)\cdots(\beta+k-1)}{(n+2k)!} A_{n,k}}, \tag{7}$$

where

$$\beta = \frac{1}{2}(n+1) \quad \text{and} \quad A_{n,k} = \sum_{\substack{i_0+\ldots+i_n=k \\ i_l \geq 0}} \frac{(2i_0)! \cdot \ldots \cdot (2i_n)!}{i_0! \cdot \ldots \cdot i_n!} .$$

The first ten values of $d_n(\infty)$ and $d_{25}(\infty)$ are given in Table 3.

Table 3. The simplicial horoball density.

n	$d_n(\infty) \approx$
2	0.95493
3	0.85328
4	0.73046
5	0.60695
6	0.49339
7	0.39441
8	0.31114
9	0.24285
10	0.18789
25	0.00238

Theorem (Kellerhals [12,28])**.** *Let $n \geq 2$, and let $Q = \mathbb{H}^n/\Gamma$ be a hyperbolic n-orbifold with $m \geq 1$ cusps. Then,*

$$\mathrm{vol}_n(Q) \geq m \cdot c_n\,, \quad \textit{where} \quad c_n = \frac{\mathrm{vol}^{\circ}_{n-1}(B(\frac{1}{2}))}{(n-1) \cdot \varphi_{n-1} \cdot \delta_{n-1} \cdot d_n(\infty)} . \tag{8}$$

As an example, by using the data of the Leech lattice in (8), the volume of a 25-dimensional hyperbolic orbifold with $m \geq 1$ cusps can be bounded from below as follows.

$$\mathrm{vol}_{25}(Q) \geq m \cdot \frac{1}{2^{49} \times 3^{10} \times 5^4 \times 7^2 \times 11 \times 13 \times 23} \cdot \frac{1}{d_{25}(\infty)} \approx m \cdot 1.25488 \times 10^{-25} . \tag{9}$$

The questions about the explicit value and the realisation of the minimal volume ν_n as $\mathrm{vol}_n(\mathbb{H}^n/\Gamma_\circ)$ have only partial answers so far. By the above theorem, one deduces the bound $\nu_n \geq c_n$ for $n \geq 2$, which is a key ingredient in answering the question for $n \leq 9$. In fact, the classical results for the dimensions $n = 2$, due to Siegel, and $n = 3$, due to Meyerhoff, were extended by Hild-Kellerhals [28] for $n = 4$ and by Hild [13,14] for $n \leq 9$, with the consequence that, for these dimensions, the unique covolume minimising groups $\Gamma_\circ \subset \mathrm{Isom}\,\mathbb{H}^n$ are given by certain hyperbolic Coxeter groups (up to index two in dimension $n = 7$). For a survey, see [1]. It turns out that all these groups $\Gamma_\circ \subset \mathrm{Isom}\,\mathbb{H}^n$, $n \leq 9$, are arithmetic and related to a tessellation of $\mathbb{H}^n$ by a 1-cusped Coxeter simplex.

3.1. The Arithmetic Case

In view of the situation just described and when looking to dimensions $n \geq 10$, it makes sense to study the (proper) subset $\mathcal{V}_n^a \subset \mathcal{V}_n$ of all volumes of orientable cusped hyperbolic n-orbifolds with *arithmetic* fundamental groups and to ask corresponding questions about the minimal element in $\mathcal{V}_n^a$, denoted by $\nu_n^a > 0$.

For an arbitrary dimension n, there is the standard arithmetic group $\mathrm{PO}(\mathrm{I}_{n,1})$ of automorphisms that leave the form q_n invariant. This group provides a first candidate for small volume. As already mentioned, for $n \leq 19$, the group $\mathrm{PO}(\mathrm{I}_{n,1})$ is reflective and can be written as the semi-direct product of its cofinite maximal reflection subgroup and the symmetry group $\mathrm{Sym}(P_n)$ of its (fundamental) Coxeter polyhedron P_n. For the covolume of $\mathrm{PO}(\mathrm{I}_{n,1})$, one has the following result for n even.

Theorem (Ratcliffe-Tschantz [27], Theorem 22)**.**

$$|\chi(\mathrm{PO}(\mathrm{I}_{n,1}))| = \left(1 \pm 2^{-\frac{n}{2}}\right) \prod_{k=1}^{\frac{n}{2}} |\zeta(1-2k)| \tag{10}$$

with a plus sign if $n \equiv 0, 2 \bmod 8$ *and the minus sign if* $n \equiv 4, 6 \bmod 8$.

3.1.1. Even Dimensions

In a much more general context, Belolipetsky and Emery (see [2–5] and [9]) successfully exploited a relevant structural result of Prasad and determined the explicit value of ν_n^a for the cases of orientable cusped arithmetic orbifolds of even dimensions $n \geq 4$ and of odd dimensions $n \geq 5$, respectively (notice that non-compactness is not a constraint in their works). In particular, for even dimensions, there is the following result in terms of the Euler characteristic.

Theorem (Belolipetsky [2,3])**.** *For each dimension $n = 2r \geq 4$ there is a unique orientable cusped arithmetic hyperbolic n-orbifold Q_n of minimal volume. It has Euler characteristic:*

$$|\chi(Q_n)| = \frac{\alpha(r)}{2^{r-2}} \prod_{k=1}^{r} |\zeta(1-2k)|\,,$$

where $\alpha(r) = 1$ *if* $r \equiv 0, 1 \pmod 4$, *and* $\alpha(r) = (2^r - 1)/2$ *if* $r \equiv 2, 3 \pmod 4$.

While the evaluation of Belolipetsky's theorem for even dimensions $4 \leq n \leq 8$ coincides with the previously mentioned (and more explicit) results of Kellerhals and Hild (see [28] and [14]), it yields the following Table 4 for even dimensions $10 \leq n \leq 18$.

Table 4. The values $|\chi(Q_n)|$ for even dimensions $10 \leq n \leq 18$.

$n \geq 10$	10	12	14	16	18
$\lvert\chi(Q_n)\rvert$	$\frac{10^{-2}}{919683072}$	$\frac{691\times 10^{-3}}{191294078976}$	$\frac{87757\times 10^{-3}}{289236647411712}$	$\frac{2499347\times 10^{-4}}{236017104287956992}$	$\frac{109638854849\times 10^{-4}}{678029937198442846617 6}$

Let us compare the results in Belolipetsky's theorem with the values $\chi(\Gamma_n)$, $10 \leq n = 2r \leq 18$, and $\chi(\Gamma_{18}^L)$ obtained in Section 2.6. Some of the Coxeter group examples presented in Section 2.3 have Coxeter graphs admitting a non-trivial symmetry group $\mathcal{S}_k$ of order k, say, which corresponds to the symmetry group of the same order of the associated Coxeter polyhedron. By extending the Coxeter group by $\mathcal{S}_k$, we pass to a group of $1/k$-times the covolume of the original group. Furthermore, since reflections are orientation reversing isometries, we need to pass to the index two orientation preserving subgroup. By taking into account the uniqueness property in Belolipetsky's result, we can deduce the following explicit volume minimality result.

Proposition 1. *Let n be even with $n \in \{10, 12, 14, 16, 18\}$. Then, the unique orientable cusped arithmetic hyperbolic n-orbifold $Q_n = \mathbb{H}^n/\Delta_n$ is given by the action on $\mathbb{H}^n$ of the index two orientation preserving subgroup Δ_n of the group $\Theta_n \subset \mathrm{Isom}\,\mathbb{H}^n$ given by Table 5.*

Table 5. The groups Θ_n.

n	Group Θ_n	$\lvert\chi(\Theta_n)\rvert$	Related Coxeter Graph
10	Γ_{10}	$\frac{10^{-2}}{1839366144}$	Figure 2
12	Γ_{12}	$\frac{691\times10^{-3}}{38258815752}$	Figure 3
14	$\Gamma_{14}\star\mathcal{S}_2$	$\frac{87757\times10^{-3}}{578473294823424}$	Figure 5
16	$\Gamma_{16}\star\mathcal{S}_2$	$\frac{2499347\times10^{-4}}{4720342085 75913984}$	Figure 7
18	$\Gamma_{18}^{L}\star\mathcal{S}_6$	$\frac{109638854849\times10^{-4}}{1356059874396885 6932352}$	Figure 8

3.1.2. Odd Dimensions

For odd dimensions $n \leq 9$, the results of Hild provide a complete picture about minimal volume cusped hyperbolic n-orbifolds, arithmetically defined or not, including proofs for uniqueness, a presentation of the fundamental group, and the value of ν_n. The orbifolds are closely related to Coxeter simplices, which do not exist for dimensions $n \geq 10$ (see Example 1 and Table 1). Combinatorially very close are pyramids over a product of *two* simplices of positive dimensions, which have been studied and classified in the Coxeter case by Tumarkin (see Section 2.3). These groups are generated by $n+2$ reflections, they are all non-compact and exist in $\operatorname{Isom}\mathbb{H}^n$ for all $4 \leq n \leq 17$ with $n \neq 14, 15, 16$. By Vinberg's arithmeticity criterion (see Section 2.4), one verifies easily their arithmeticity when $n \geq 11$. According to the corresponding commensurability classification performed in [7], one has five Coxeter pyramid groups in $\operatorname{Isom}\mathbb{H}^{11}$ falling into two commensurability classes, three Coxeter pyramid groups in $\operatorname{Isom}\mathbb{H}^{13}$ forming one commensurability class, and finally the single Coxeter pyramid group $\Gamma_* \subset \operatorname{Isom}\mathbb{H}^{17}$ that is closely related to the automorphism group of the even unimodular group $\mathrm{PO}(\mathrm{II}_{17,1})$ (see Example 2). Among the five arithmetic Coxeter pyramid groups $\operatorname{Isom}\mathbb{H}^{11}$, which fall into two commensurability classes, the group Γ_{11} given by the graph in Figure 9 has smallest covolume, and among the three commensurable Coxeter pyramid groups in $\operatorname{Isom}\mathbb{H}^{13}$, the group Γ_{13} given by Figure 11 has smallest covolume (see [7] and [32]).

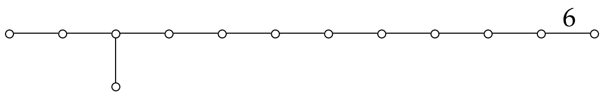


Figure 9. The Coxeter pyramid group $\Gamma_{11} \subset \operatorname{Isom}\mathbb{H}^{11}$.

In order to identify explicitly—if possible—the minimal volume orientable cusped arithmetic hyperbolic n-orbifolds for $n \geq 11$ odd, we provide details of the corresponding result of Belolipetsky and Emery (see Section 3.1.1).

Theorem (Belolipetsky, Emery [4,5])**.** *For each dimension $n = 2r-1 \geq 5$, there is a unique orientable arithmetic cusped hyperbolic n-orbifold Q_n of minimal volume. Its volume is given by the following formula.*

(1) If $r \equiv 1 \pmod 4$:

$$\mathrm{vol}_n(Q_n) \quad = \quad \frac{1}{2^{r-2}}\,\zeta(r)\prod_{k=1}^{r-1}\frac{(2k-1)!}{(2\pi)^{2k}}\zeta(2k)\,;$$

(2) If $r \equiv 3 \pmod 4$:

$$\mathrm{vol}_n(Q_n) \quad = \quad \frac{(2^r-1)(2^{r-1}-1)}{3\cdot 2^{r-1}}\,\zeta(r)\prod_{k=1}^{r-1}\frac{(2k-1)!}{(2\pi)^{2k}}\zeta(2k)\,;$$

(3) If r is even:

$$\mathrm{vol}_n(Q_n) = \frac{3^{r-1/2}}{2^{r-1}} L_{\ell_1|\mathbb{Q}}(r) \prod_{k=1}^{r-1} \frac{(2k-1)!}{(2\pi)^{2k}} \zeta(2k)\,, \quad \textit{where} \quad \ell_1 = \mathbb{Q}(\sqrt{-3})\,.$$

In [9], Emery described in more detail the fundamental group Δ_n of the orientable arithmetic cusped orbifold Q_n of minimal volume as follows. For $n \equiv 1 \pmod 8$, the group $\mathrm{PSO}(\mathrm{II}_{n,1})$ is conjugate to Δ_n in $\mathrm{Isom}\,\mathbb{H}^n$, while for $n \equiv 5 \pmod 8$ the group $\mathrm{PSO}(\mathrm{I}_{n,1})$ is conjugate to a subgroup of index 3 of Δ_n in $\mathrm{Isom}\,\mathbb{H}^n$. For $n \equiv 3 \pmod 4$, the group Δ_n is commensurable to the group $\mathrm{PO}(f_3;\mathbb{Z})$ of integral automorphisms of the Lorentzian form of signature $(n,1)$ given by:

$$f_3(x) = x_1^2 + \ldots + x_n^2 - 3\,x_{n+1}^2\ . \tag{11}$$

By a result of Mcleod [33], the group $\mathrm{PO}(f_3;\mathbb{Z})$ is reflective for $n \leq 13$. As an extension of their work for $\mathrm{PO}(n,1;\mathbb{Z})$ to the group $\mathrm{PO}(f_d;\mathbb{Z})$, Ratcliffe and Tschantz determined the covolumes of the groups $\mathrm{PO}(f_3;\mathbb{Z})$ and, for each $n \equiv 3 \pmod 4$, they computed furthermore the commensurability ratio $\kappa_n \in \mathbb{Q}$ of Δ_n and $\mathrm{PO}(f_3;\mathbb{Z})$ showing that $\kappa_n \neq 1$ (see [34], (35)).

In view of these results and the knowledge of hyperbolic Coxeter group candidates in $\mathrm{Isom}\,\mathbb{H}^n$ for $n = 11, 13$ and $n = 17$, we provide the following new characterisation of $Q_n = \mathbb{H}^n/\Delta_n$ for $n = 11, 13$ and mention briefly the known result of Emery [9] in the case $n = 17$.

The case $n = 11$. Since $11 \equiv 3 \pmod 4$, the group Δ_{11} of minimal covolume is commensurable to the group $\mathrm{PO}(f_3;\mathbb{Z})$, whose cofinite maximal reflection subgroup Γ was described by Mcleod. More precisely, the Coxeter graph of Γ, given by Figure 10, has 15 nodes and shows a two-fold symmetry. Denote by $P \subset \mathrm{Isom}\,\mathbb{H}^{11}$ the Coxeter polyhedron of Γ.

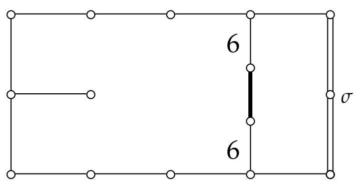


Figure 10. Mcleod's Coxeter group $\Gamma \subset \mathrm{Isom}\,\mathbb{H}^{11}$.

In particular, by the result ([34], Table 1), one gets the value

$$\mathrm{covol}_{11}(\mathrm{PO}(f_3;\mathbb{Z})) = \frac{13 \times 31}{2^{25} \times 5 \times 7 \times 11 \times \sqrt{3}} \cdot L(6,-3)\ , \tag{12}$$

where $L(s,D)$ is Dirichlet's L-function according to ([34], (12)), as well as $\kappa_{11} = \frac{1}{4}\,(2^5-1)\,(2^6+1) = \frac{2015}{4}$ (see [34], Section 7). This implies that $\mathrm{vol}_{11}(P) = 2 \cdot \mathrm{covol}_{11}(\mathrm{PO}(f_3;\mathbb{Z}))$ and that $\mathrm{covol}_{11}(\Delta_{11}) = \frac{2}{2015}\,\mathrm{vol}_{11}(P)$.

Now, consider the group Γ_{11} which has smallest covolume among all Coxeter pyramid groups in $\mathbb{H}^{11}$ and let P_{11} be its Coxeter pyramid. Based on an observation of Tschantz [10] when comparing the corresponding Coxeter graphs, there is a close combinatorial relation between the polyhedron P associated to Mcleod's Coxeter group Γ and the Coxeter pyramid P_{11}. In fact, pass to the double P_σ of the polyhedron P by reflecting it in the bounding hyperplane depicted by σ in the graph of Figure 10. Then, the polyhedron P_σ is bounded by 16 hyperplanes. Reflect recursively the pyramid P_{11} in its facets while staying inside the polyhedron P_σ. The image pyramids match along their facets or line up with the facets of P_σ. It takes exactly 4030 copies of the pyramid P_{11} to fill P_σ. As a consequence and by (12), one obtains:

$$\begin{aligned} \mathrm{vol}_{11}(P_{11}) &= \frac{2}{4030}\,\mathrm{vol}_{11}(P) = \frac{4}{4030}\,\mathrm{covol}_{11}(\mathrm{PO}(f_3;\mathbb{Z})) \\ &= \frac{1}{2^{24}\times 5^2\times 7\times 11\times\sqrt{3}}\cdot L(6,-3) \approx 1.760074651\times 10^{-11}\,. \end{aligned} \tag{13}$$

Putting everything together, one deduces that $\mathrm{covol}_{11}(\Delta_{11}) = 2\cdot\mathrm{covol}_{11}(\Gamma_{11})$. By passing to the index two subgroup Γ_{11}^+ of orientation preserving isometries in the Coxeter pyramid group Γ_{11}, one finally obtains the following result.

Proposition 2. *The orientable arithmetic cusped hyperbolic orbifold Q_{11} of minimal volume is the quotient of $\mathbb{H}^{11}$ by the rotation subgroup Γ_{11}^+ of Γ_{11}. The value $2\,v_{11}^a$ is given by (13).*

The case $n = 13$. Since 13 satisfies $n \equiv 5 \pmod 8$, the fundamental group Δ_n of the orientable arithmetic cusped orbifold Q_n of minimal volume is commensurable to the special unimodular group $\mathrm{PSO}(\mathrm{I}_{13})$, with ratio of their covolumes equal to 3 (see [9], Proposition 5). By the result [27], Theorem 6, of Ratcliffe and Tschantz mentioned in Remark 1, the covolume of $\mathrm{PSO}(\mathrm{I}_{13})$, being of index two in $\mathrm{PO}(\mathrm{I}_{13})$, can be expressed as follows.

$$\begin{aligned} \mathrm{covol}_{13}(\mathrm{PSO}(\mathrm{I}_{13,1})) &= (2^7-1)\times(2^6-1)\prod_{k=1}^{6}\frac{|B_{2k}|}{8k}\cdot\zeta(7) \\ &= \frac{127}{2^{28}\times 3^6\times 5^3\times 7\times 11\times 13}\,\zeta(7) \approx 3.613942699\times 10^{-12}\,. \end{aligned} \tag{14}$$

Since the group $\mathrm{PO}(\mathrm{I}_{13})$ is known to be reflective and equal to the Coxeter pyramid group Γ_{13}, with the graph given by Figure 11, we can deduce the following result.

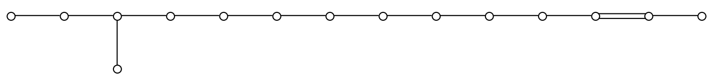


Figure 11. The Coxeter pyramid group $\Gamma_{13} \subset \mathrm{Isom}\,\mathbb{H}^{13}$.

Proposition 3. *The orientable arithmetic cusped hyperbolic 13-orbifold Q_{13} of minimal volume is the quotient of $\mathbb{H}^{13}$ by the rotation subgroup Γ_{13}^+ of Γ_{13}. Its volume v_{13}^a is given by (14).*

The case $n = 17$. Let us finish by mentioning the result [9], Theorem 2, of Emery. It states that for $n \equiv 5 \pmod 8$, the minimal volume orientable arithmetic cusped hyperbolic n-orbifold Q_n is the quotient space $\mathbb{H}^n/\mathrm{PSO}(\mathrm{II}_{n,1})$. For $n = 17$, the group $\mathrm{PO}(\mathrm{II}_{17,1})$ is reflective and is the semi-direct product of the reflection group Γ_* with the symmetry group $\mathcal{S}_2$ of P_*, where P_* is Tumarkin's Coxeter pyramid with graph given in Figure 1 and described in Example 2. By exploiting the theorem above, one gets the following volume identification (see [9], Corollary 3, Corollary 4).

$$\begin{aligned} \mathrm{vol}_{17}(P_*) &= \frac{1}{2}\cdot\mathrm{covol}_{17}(\mathrm{PSO}(\mathrm{I}_{17,1})) = \frac{691\times 3617}{2^{38}\times 3^{10}\times 5^4\times 7^2\times 11\times 13\cdot 17}\cdot\zeta(9) \\ &\approx 2.072451981\times 10^{-18}\,. \end{aligned} \tag{15}$$

As mentioned by Emery in [9], Section 3, the space $\mathbb{H}^{17}/\mathrm{PSO}(\mathrm{II}_{17,1})$ has minimal volume among *all* orientable arithmetic hyperbolic n-orbifold Q_n, compact or not, for $n \geq 2$. This means that $v_n^a > v_{17}^a = \mathrm{covol}_{17}(\mathrm{PSO}(\mathrm{I}_{17,1}))$ for *all* $n \geq 2$, $n \neq 17$.

Final remarks.

(1) *When looking at realisations of orbifolds with volumes equal to the minimal values v_n^a for $n \leq 18$, there remains a need to study the case $n = 15$ and to look for a candidate in the commensurability class of $PO(f_3;\mathbb{Z})$ that is conjugate to the fundamental group of the minimal volume hyperbolic orbifold of dimension 15.*

(2) *It is an interesting but difficult question whether, and to what extent, non-arithmetic considerations can perturb the picture described in Section 3 in such a way that $v_n^a > v_n$ for some $n > 3$.*

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